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TRANSIENTS AND ARNOLD TONGUES FOR SYNCHRONIZED ELECTRONIC FIREFLIES

N. Rubido[14], *C. Cabeza*[1], *S. Kahan*[2], *G. M. Ramírez Ávila*[3], *A. C. Martí*[1]

[1] Instituto de Física, Facultad de Ciencias, Universidad de la República, Montevideo, Uruguay
[2] Instituto de Física, Facultad de Ingeniería, Universidad de la República, Montevideo, Uruguay
[3] Instituto de Investigaciones Físicas, Universidad Mayor de San Andrés, La Paz, Bolivia
[4]nrubido@fisica.edu.uy

Abstract: Fireflies constitute a paradigm of pulse-coupled oscillators. This pulse coupling form is extensively common in Biology —the chirp of crickets, pacemaker cells firing and luminescent algae *Gonyaulax*, among many—. The study of how pulse-coupled oscillators achieve synchrony is important due to experimental observations of synchronous neural firing patterns of various mammalians, insects and reptilian species. In order to tackle the problems related to synchrony of pulse-coupled oscillators, a Light-Controlled Oscillator (LCO) model is presented. LCOs constitute unidimensional relaxation oscillators described by two distinct timescales meant to mimic *Pteroptyx malaccae* fireflies in a simple fashion, with great parameter malleability and easy experimental implementation. Dynamical results dealt range from transient behaviours for different coupling configurations and intensities, to stable states of arbitrary order. Furthermore, analytical expressions regarding situation are also exhibited. Construction of return maps reveal stability issues, bifurcations of fixed points as control parameters are tunned and as the number of oscillators involved is increased. Numerical simulations complement all studies.

keywords: synchronization, pulse-coupling oscillators, coupling configurations

1. INTRODUCTION

Synchronization consists in an adjustment of rhythms among self-sustained systems due to a weak coupling [1, 2] that may act in different manners. This phenomena manifests in systems of very different nature. In particular, pulse-coupled oscillators and integrate-and-fire models have long attracted the attention of several scientists, though, significant progress in its analysis has not been made when the oscillators are not identical. As a consequence, mainly numerical results have been reported. Nevertheless, Strogatz and Mirollo [3] proved that for certain conditions, these type of coupled oscillators always synchronize.

Experimental works that deal with synchronization problems are usually based in complex models in which transient times cannot be easily interpreted. Transients are usually elucidated for simpler models; configurations of phase-locked loops [4] and Light-Controlled Oscillators (LCOs) [3, 5, 6] constitute examples of them.

As a complement to dynamical considerations, we present an analysis due to self-similarity of synchronization times as a function of initial conditions curves for two coupled LCOs. These analysis shows scaling laws that result in the determination of critical exponents and universal behaviours that make the transient times dependent only on initial conditions and the adjacency matrix form, despite coupling intensity. Furthermore, the studies related to Master-Slave (MS) and Mutual-Interaction (MI) configurations reveal the known existence of a single attracting limit cycle as well as a repelor that corresponds to a single unstable initial state. The ability to calculate Floquet exponents for this cycles are based in a simple transformation of coordinates.

On the other hand, synchronous regimes of arbitrary order are observed, and the whole family of synchronization regions are plotted for MS and MI configurations in a numerical and experimental fashion. A return map is generated by the construction of a Poincaré section that allows to obtain some analytical results for the MS configuration. Results in this matter exhibit fixed points and stable states for arbitrary $(1:n)$ states with remarkable consequences on the tongues stability and form. Further analysis of stable regions is carried for different configurations of three LCOs, where experimental and numerical results exhibit that depending on the adjacency matrix, phase bifurcations appear as a consequence of frequency detuning.

2. LIGHT-CONTROLLED OSCILLATORS

LCOs are piecewise-linear one-dimensional oscillators, set to oscillate in a spiking form. The electronics consists on a dual RC circuit that shares the capacitor and an LM555 chip. This last element chooses the corresponding stage by stablishing well defined thresholds. Being V_{cc} the voltage source, the threshold for the charging (discharging) stage is $2V_{cc}/3$ ($V_{cc}/3$). Coupling is achieved by IR diodes and photosensors. Coupling strengths are set by placing LCOs at different distances.

The dynamical model that describes LCOs corresponds to the following set of differential equations [5, 6]:

$$\dot{V}_i(t) = \lambda_i \left[V_{cc} - V_i(t)\right] \epsilon_i(t) - \gamma_i V_i(t) \left[1 - \epsilon_i(t)\right] + \beta \sum_{j=1,\, j\neq i}^{N} \delta_{ij} \left[1 - \epsilon_j(t)\right],\ i = 1, \ldots, N, \qquad (1)$$

where V_i is the i-th LCO voltage, β gives account of the coupling strength, δ_{ij} is the adjacency matrix element, and $\epsilon_i(t)$ is a variable created to represent the oscillator stage —takes the value 1 (charging stage) or 0 (discharging stage)—. The parameter λ_i (γ_i) is the inverse characteristic time scale for the charging (discharging) stage and is related to those of the LCO. The action of the coupling results in a raise of the asymptotic level of the capacitor stages (V_{cc} and 0 increase for charge and discharge, respectively).

In the case of an isolated LCO the dynamic is naturally represented in the circle S^1. This state space has an injective and a dissipative part of similar length. Through the injective part, the LCO charges during a time T_λ, while being in the dissipative part a fast discharge occurs lasting T_γ. Every time any LCO achieves a threshold a new initial condition for the global system is generated. Nevertheless, the contraction of S^1 into a line segment proves to be fruitful when representing N LCOs, as the state space then transforms into an N-dimensional cube given by $[V_{cc}/3,\, 2V_{cc}/3]^N$ where $V_{cc} = 9\ V$. Represented in a N-dimensional torus, this manifold has 2^N sections that represent different flux behaviours and timescale ratios, according to the 2^N stage possibilities. In the N-dimensional cube, this sections merge into an abrupt flux change of direction and magnitude every time a threshold is achieved, like a ball bouncing in a box with reactive walls.

3. TRANSIENTS AND STABLE STATES

Synchronization times were calculated by means of the Poincaré section based, for instance, on LCO_1 upper threshold. Then, a phase point transformation (PPT) is constructed:

$$\phi_i(t) = 2\pi\ (t - t_k) \,/\, (t_{k+1} - t_k)\ ,\ \ \forall t \in [t_k, t_{k+1}]\,, \quad (2)$$

where ϕ_i the i-th oscillator phase ($i = 2, \ldots, N$), and t_k is the k-th time that LCO_1 achieves the upper threshold. Synchronization between LCO_1 and LCO_i is achieved whenever ϕ_i tends to a constant.

Due to Eq. (1), synchronization times t_{synch} will depend on the coupling configuration and on initial conditions. For two coupled LCOs: $t_{synch} = f\,(V_{init},\, \beta,\, \delta_{ij})$. Experimental and numerical results show that maintaining the same configuration self-similar curves appear for this dependence when β is tuned. Therefore, scaling laws are calculated.

Cubic state spaces exhibit more directly synchronization properties. In particular, the limit cycle of full synchronization corresponds to the cube diagonal, thus $45°$ plane rotations were used in order to generate orthogonal and parallel variables that allowed Floquet exponent calculations. Whenever the synchronization manifold has a higher dimension in this system, it is due to phase-lags. Thus, correlation functions help to withdraw this lags and maintain the transformation properties for characteristic exponent calculations.

Regarding stable states, there are several locking possibilities according to the frequency ratio of the oscillators. Whenever this ratio is near a rational number, synchronization $(m : n)$ can be achieved.

Time locking possibilities for $(1 : n)$ states for a MS configuration result in analytical constrains for the Master charging time (T_{λ_M}) as a function of Slave characteristic times. The construction of the return map: $V_{n+1} = f(V_n)$, where V_n is the Slave LCO voltage at the start of the n-th interaction, shows a piecewise flux of six parts. A stable fixed point occurs whenever Master LCO begins to act on Slave LCO charge and finishes on Slave discharge. The unstable fixed point lies in the part of the map where interactions occur through discharges and end on the charging state. In order to achieve synchronization, two pieces of the map with negative slope detach the flux in two ways and connect these last situations. Near the edges of these tongues the fixed points merge in a saddle-node type bifurcation and disappear. Surprisingly, all $(1 : n)$ tongues maintain these characteristics.

When dealing with three LCOs, transients are solved by calculating the Floquet exponents of the orthogonal $3D$ cube diagonal directions. On the other hand, as coupling configurations can take 15 different forms (corresponding to the six independent entries of the adjacency matrix), stable states are analyzed taking constant β and detuning LCO periods for each configuration. Results exhibit phase bifurcations for some of these configurations when changes in a particular T_λ is made, detuning the LCOs involved.

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