Eur. Phys. J. Special Topics
© EDP Sciences, Springer-Verlag 2016
DOI: 10.1140/epjst/e2016-60033-5

THE EUROPEAN PHYSICAL JOURNAL SPECIAL TOPICS

Review

Unraveling the primary mechanisms leading to synchronization response in dissimilar oscillators

Gonozalo Marcelo Ramírez-Ávila[1,a] and Juergen Kurths[2,3,4,5]

[1] Instituto de Investigaciones Físicas, Casilla 8635 Universidad Mayor de San Andrés, La Paz, Bolivia
[2] Institut für Physik, Humboldt-Universität zu Berlin, Robert-Koch-Platz 4, 10115, Berlin, Germany
[3] Potsdam Institut für Klimafolgenforschung, PO Box 60 12 03, 14412 Potsdam, Germany
[4] Institute for Complex Systems and Mathematical Biology, University of Aberdeen, Aberdeen AB24 3FX, United Kingdom
[5] Department of Control Theory, Nizhny Novgorod State University, 606950 Nizhny Novgorod, Russia

Received 19 February 2016 / Received in final form 15 June 2016
Published online xx xxx 2016

Abstract. We study how the phenomenon of response to synchronization arises in sets of pulse-coupled dissimilar oscillators. One of the sets is constituted by oscillators that can easily synchronize. Conversely, the oscillators of the other set do not synchronize. When the elements of the first set are not synchronized, they induce oscillation death in the constituents of the second set. By contrast, when synchronization is achieved in oscillators of the first set, those of the second set recover their oscillatory behavior and thus, responding to synchronization. Additionally, we found another interesting phenomenon in this type of systems, namely, a new control of simultaneous firings in a population of similar oscillators attained by means of the action of a dissimilar oscillator.

1 Introduction

Synchronization is one of the most widespread phenomena in nature and in man-made systems. It has been studied from a formal viewpoint in systems that are mainly related to vibrational mechanics [1], where this phenomenon is manifested when the oscillating or rotational systems start moving with the same multiple or commensurable frequencies in the presence of even very weak interactions [2]. In other words, synchronization is defined as the adjustment of rhythms of two or more oscillators due to their weak interaction [3]. The observation of certain features in synchronous systems has motivated the search of a unifying framework for synchronization and the specification of a diversity of phenomena such as generalized and identical synchronization, phase synchronization, and lag synchronization [4]. Several works

[a] e-mail: gramirez@ulb.ac.be

dealing with synchronization have been conducted in physics [5,6], chemistry [7], biology [8], and applications were developed in communication systems and control [9], in networks [10] particularly those concerned to power grids [11], neural [12,13] and social [14] ones. The scope of synchronization also reaches unusual topics such as musicology [15]. General concepts and applications might be found in the vast existing literature on synchronization [3,16–20], where the Kuramoto [21,22] and the integrate-and-fire [23–25] oscillators constitute paradigmatic models describing this kind of phenomena. Alongside the concept of synchronization, it is very important to take into account multistability [26], a feature causing different patterns of synchronization and/or differing transient (synchronization time). Another interesting phenomenon occurring in coupled oscillators is the emergence of amplitude and oscillation death [27] that may be manifested totally or partially [28], produced by means of time-delay [29] or by strong coupling [30]. The mechanisms of oscillation quenching is explained widely in [31], and different transitions from amplitude to oscillation death in [32].

Synchronization of the flashing behavior of certain firefly species aroused the curiosity of biologists [33–36], and also the interest of mathematicians and physicists that attempted to model this behavior [37–39]. Since the work of Buck and Case [40] it is well-known the possibility to influence on the rhythmic behavior of fireflies. Additionally, an astonishing finding issued from experiments carried out with virtual males and a real American species *Photinus carolinus* female, has shown that males' synchronization is associated with the female's response [41]. The latter improved the knowledge concerning the synchronous behavior of fireflies, enhancing the fact that both males and females participate actively in the courtship. In other words, when referring to fireflies courtship, we must consider the females' response to males' synchronization. A first attempt to explain response to synchronization [42] has been made using light-controlled oscillators (LCOs) [39] with dissimilar features as prototype to model male and female behavior separately and/or when they are interacting. The model described in [42] not only reproduces the experimental results shown in [41] but it is also capable to predict more complex and realistic situations.

In general, the observation of natural phenomena leads to find an explanation of those by formulating models. Furthermore, on some occasions, these phenomena are the source of inspiration in the implementation of new technologies [43] or specific applications [44], generating the possibility to extend and generalize the concepts to other situations. Precisely, in this work we try to extend the response to synchronization observed in fireflies to any system of dissimilar pulse-coupled oscillators by means of a physical explanation of how the response to synchronization takes place.

The four main concepts used in this research are synchronization, oscillation death, multistability, and dissimilar pulse-coupled oscillators, which are developed throughout the paper. In Sect. 2, we introduce the equations of dissimilar oscillators and basic intrinsic aspects thereof are explained. The mechanisms yielding the response to synchronization are detailed in Sect. 3. We discuss in Sect. 4 the main aspects related to how a group of synchronous oscillators can give rise to a response of another group of oscillators and we sum up the most important features found in the studied systems. Finally, in Sect. 5, we summarize the results giving conclusions and perspectives.

2 Model and main features of the oscillators

Rhythmic behavior of fireflies is one of the most invoked phenomena when talking about natural systems with the ability to synchronize. The synchronous behavior of male fireflies of some species found its functional interpretation [45] in the courtship display exhibited by these insects [46]. As stated in Sect. 1, we base our work on a phenomenon observed in *Photinus carolinus*, a firefly species exhibiting the response to synchronization. The females' response is an act that follows the courtship display performed by the males. In this paper, we consider the same behavior manifested by fireflies, i.e., response to synchronization but focusing only on the dynamical aspects of the oscillators, regardless of the biological ones.

2.1 Individual oscillators

Signals of two types of oscillators are represented in Fig. 1 and also the terminology used in the description of bursting oscillators [47]. The first type (Fig. 1a) fires a burst of n_f spikes during the active phase, followed by a quiescent or silent time T_s, a parameter that remains constant even when the oscillators are coupled. The other type has just one spike in its fast firing (discharging) process T_d which is preceded by a long lasting charging process T_c and followed by a silent time T_s (Fig. 1b). We define the interburst period or the duration of a phrase T_p as the complete cycle comprising the active phase and the silent time. Consequently, the active phase takes $n_f(T_c + T_d) = Tp - Ts$. Both types of oscillators are individually considered as relaxation oscillators due to their intrinsic characteristics of having two different time scales, i.e., within each cycle there is an integrating (slow) process followed by a firing (fast) process. Each process ends at its own threshold, being the lower and the upper thresholds at $V^{\text{lower}} = V_M/3 = 3$ and $V^{\text{upper}} = 2V_M/3 = 6$ respectively. We take these threshold values in connection with the experimental aspects related to the LCO, namely, the oscillator serving as the basis of the model stated in Eq. (1). Note that we take V_M=9 which is the considered value from an experimental point of view and related to the value of a voltage source.

The equations describing the dynamical variable V_i of each oscillator i are given by:

$$\frac{\mathrm{d}V_i(t)}{\mathrm{d}t} = \frac{\ln 2}{T_{ci}} \left(V_{Mi} - V_i(t)\right) \varepsilon_i(t) - \frac{\ln 2}{T_{di}} V_i(t) \left(1 - \varepsilon_i(t)\right), \tag{1a}$$

$$V_i(t) = \left(V_i(t) - V_i^{\text{lower}}\right) \varepsilon_i(t) + V_i^{\text{lower}}. \tag{1b}$$

As stated above, V_M is a constant that determines the lower and upper thresholds and $\varepsilon_i(t)$ is a binary variable describing the state of the ith oscillator by:

$$\begin{aligned} &\varepsilon_i(t) = 1 : \text{extinguished oscillator (charging and silent stage)} \\ &\varepsilon_i(t) = 0 : \text{fired oscillator} \qquad\qquad \text{(discharging stage).} \end{aligned}$$

The transition between the states determined by ε is described by the following relation:

$$\textbf{If} \quad V_i(t) = V_i^{\text{lower}} \quad \textbf{and} \quad \varepsilon_i(t) = 0 \quad \textbf{then} \quad \varepsilon_i(t_+) = 1; \tag{2a}$$

$$\textbf{If} \quad V_i(t) = V_i^{\text{upper}} \quad \textbf{and} \quad \varepsilon_i(t) = 1 \quad \textbf{then} \quad \varepsilon_i(t_+) = 0; \tag{2b}$$

$$\textbf{If} \quad V_i(t) = V_i^{\text{lower}} \quad \textbf{and} \quad \varepsilon_i(t) = 1 \quad \textbf{then} \quad \varepsilon_i(t_+) = 1, \tag{2c}$$

where t_+ in the condition given by Eq. (2c) is defined in the interval

$$t = [t_+(k-1)(T_p + n_f(T_c + T_d)) + \Delta\phi]$$

for every k interburst period or phrase, i.e., for every complete cycle comprising the active phase and the silent time.

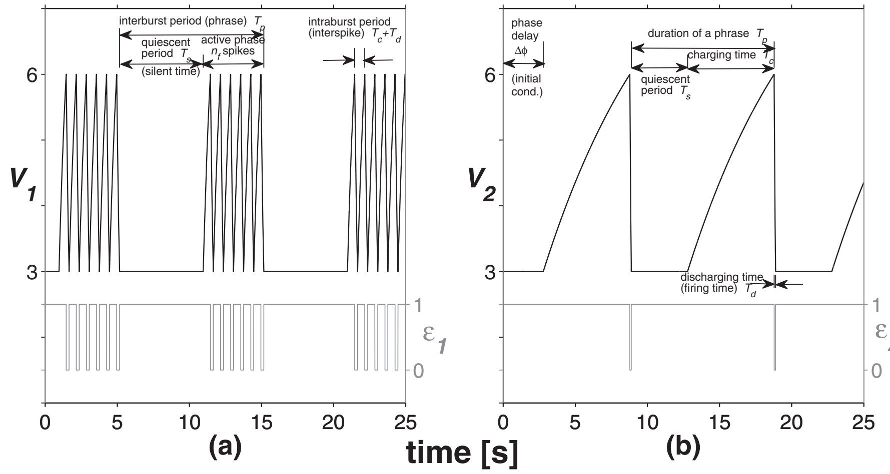

Fig. 1. Signals of the dynamic V and the binary ε variables for the two types of relaxation oscillators used in this work. They are characterized by the quiescent period T_s, the active phase with n_f spikes per burst, the interburst period or silent time T_s, the charging and the discharging times T_c and T_d respectively, the intraburst or interspike period $T_c + T_d$, the interburst period or duration of a phrase T_p, and the phase delay $\Delta\phi$ that plays the role of initial condition. (a) Bursting oscillator (BO) that in this case has the following parameter values: $T_s = 10.000$ s, $n_f = 6$, $T_c = 0.500$ s, $T_d = 0.200$ s, $T_s = 5.800$ s and $\Delta\phi = 0.603$ rad $\equiv 0.960$ s. (b) Nonbursting oscillator (NBO) having in this particular case the parameter values: $T_s = 10.000$ s, $n_f = 1$, $T_c = 6.000$ s, $T_d = 0.100$ s, $T_s = 3.900$ s and $\Delta\phi = 1.750$ rad $\equiv 2.785$ s.

2.2 Coupled oscillators

The main feature of the considered oscillators dwells on its firing process which allows a pulsatile coupling with other oscillators that can receive these pulses or spikes leading to a modification in their oscillatory dynamics. The dynamical equations describing a generic group of N coupled oscillators are:

$$\frac{\mathrm{d}V_i(t)}{\mathrm{d}t} = \frac{\ln 2}{T_{c0i}}\left(V_{Mi} - V_i(t)\right)\varepsilon_i(t) - \frac{\ln 2}{T_{d0i}} V_i(t)\left(1 - \varepsilon_i(t)\right) + \theta_i \sum_{i,j=1}^{N} \beta_{ij}(1 - \varepsilon_j(t)), \quad (3)$$

where $i, j = 1, \ldots, N$. Conditions given by Eq. (1b) and Eqs. (2), which take into account the existence of a silent time, must also be followed by Eq. (3). The quantities T_{c0i} and T_{d0i} are, respectively, the lasting time for the charge and the discharge when there is no action on the oscillator i by other oscillators. Furthermore, we consider that oscillators are mutually coupled with a coupling strength β_{ij} that represents the pulsatile action of the oscillator j spike during its discharge upon the oscillator i. Concurrently, β_{ij} are the elements of the weighted adjacency matrix of the set. A simple inspection of Eq. (3) shows that both charging and discharging stages might be modified by the effect of the coupling with other oscillator(s). The charging and the discharging times might be shortened or lengthened respectively when the pulsatile action due to the firing of other oscillator(s) takes place. The latter is determined by the value of θ that takes the values:

$$\theta = \begin{cases} 1\ , & \text{Bursting oscillator (BO)} \\ -1\ , & \text{Nonbursting oscillator (NBO)} \end{cases}$$

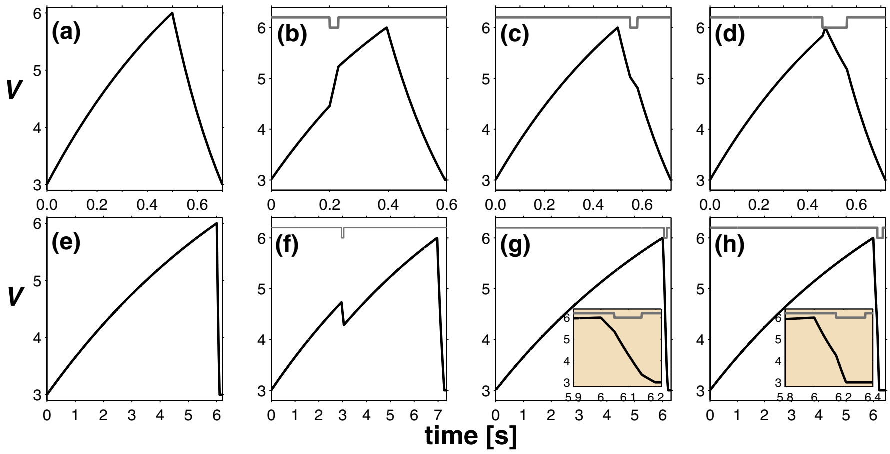

Fig. 2. Effect of an external stimulus (coupling) on signals of the dynamic variable V comprising an intraburst period $T_c + T_d$ for the two types of oscillators: BO (a)–(d) and NBO (e)–(h). The external stimulus is represented by the pulse in the upper part of the figures. The parameter values are $T_{c0} = 0.500\,\text{s}$ and $T_{d0} = 0.200\,\text{s}$ for BO; and $T_{c0} = 6.000\,\text{s}$ and $T_{d0} = 0.030\,\text{s}$ for NBO. (a) and (e) Natural (without any influence) signal. (b) and (f) stimuli $\beta = 20.0$ and $\beta = 5.0$ acting on the charging stage during 0.030 s and 0.100 s respectively. (c) and (g) stimuli $\beta = 10.0$ acting on the discharging stage during 0.030 s and 0.100 s respectively. (d) and (h) stimuli $\beta = 10.0$ acting on both charging–discharging and discharging–silent stages during 0.100 s and 0.200 s respectively. In (g) and (h) are included the insets showing with more detail the action of the stimulus and the effect on the signal.

Table 1. Effects of a stimulus acting on the different stages of the dynamic variable V, and the resulting coupling.

Type of oscillator	Stimulus acting on	Effect on V	Resulting coupling	Figure
Bursting	charging stage	shortening	excitatory	Fig. 2b
	discharging stage	lengthening	inhibitory	Fig. 2c
	both: charging and discharging stages	shortening and lengthening	excitatory and inhibitory	Fig. 2d
Nonbursting	charging stage	lengthening	inhibitory	Fig. 2f
	discharging stage	shortening	excitatory	Fig. 2g
	both: discharging and silent stages	shortening and none	excitatory and none	Fig. 2h

This factor is very important because it determines the behavior of the oscillators when stimuli are applied to them. Using Eq. (3), we can note the effect of the stimulus depending on which stage it acts as shown in Fig. 2.

Table 1 summarizes the effects on the dynamic variable V of each type of oscillator when it receives a stimulus. The effect depends on the stage of the oscillator in which the stimulus is acting.

3 Method and results

This section is devoted to the explanation of all the issues leading eventually to response to synchronization. We start by analyzing the possible coupling configurations which will give us a clearer picture of how the oscillators behave when they are coupled. For each configuration, we have several possibilities such as the case of identical

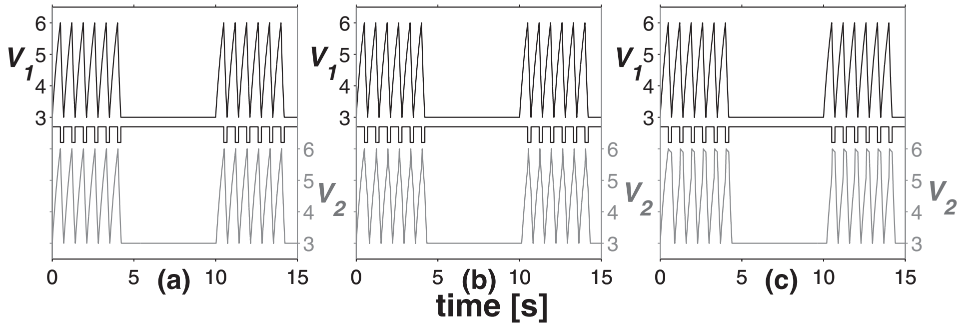

Fig. 3. Dynamic variables V_1 for the master BO and V_2 for the slave BO. The BOs are identical with parameter values: $T_{c0} = 0.500$ s, $T_{d0} = 0.200$ s and initial condition $\Delta\phi = 0.000$ s; with coupling values: (a) $\beta = 0.5$, (b) $\beta = 10.0$ and (c) $\beta = 20.0$.

Table 2. Modifications on the charging and discharging time of a slave BO compared with the "natural" ones $T_{c0} = 0.500$ s and $T_{d0} = 0.200$ s. Case of identical oscillators.

Coupling strength β	Charging time T_c [s]	% change on T_c ($\%\Delta T_c$)	Discharging time T_d [s]	% change on T_d ($\%\Delta T_d$)
0.5	0.495	−1.0	0.205	2.5
10.0	0.412	−17.6	0.288	44.0
20.0	0.343	−31.4	0.357	78.5

and nonidentical oscillators, the variation of the initial conditions and the coupling strength.

3.1 Master (BO)–slave (BO) configuration

Firstly, we consider three examples of identical oscillators represented in Fig. 3. From Fig. 3, we observe a phase delay of the slave BO, due to the fact that its charging time is shortened but the discharging time is lengthened in a greater proportion with respect to the "natural" values $T_c = 0.500$ s and $T_d = 0.200$ s as summarized in Table 2. We observe that the intraburst period $(T_c + T_d)$ remains constant, whereas for strong coupling, the discharging time becomes greater than charging time $(T_d > T_c)$ which modifies strongly the shape of the signal as shown in Fig. 3c. The results above were obtained by using the same initial conditions. A phase delay of the slave BO is still present when the initial conditions are different.

For nonidentical oscillators, we also consider three examples represented in Fig. 4, where we note that there is also a phase delay of the slave BO. As in the case of identical oscillators, synchronization is easily achieved and manifested by the equality of the slave's intraburst period with respect to that of the master as set out in Table 3.

3.2 Mutually coupled BOs configuration

When the BOs are mutually coupled and identical, we observe that synchronization is achieved with changes in both stages; furthermore, the period increases with respect to its "natural" value $T_0 = 0.700$ s as shown in Fig. 5 and Table 4. We observe that the BOs' period becomes larger owing to the fact that the discharging time climbs markedly with the coupling strength β. On the contrary, the charging time remains almost constant. From Fig. 5, we remark that synchronization is easily attainable

Table 3. Modifications on the charging and discharging time of a slave BO compared with the "natural" ones $T_{c01} = 0.500$ s and $T_{d01} = 0.200$ s. Case of nonidentical oscillators with $\beta = 8.0$.

Natural charging time T_{c02} [s]	Natural discharging time T_{d02} [s]	Charging time T_{c2} [s]	% change on T_{c2}	Discharging time T_{d2} [s]	% change on T_{d2}
0.480	0.220	0.405	−15.625	0.295	34.091
0.520	0.180	0.451	−13.269	0.249	38.333
0.510	0.210	0.421	−17.451	0.279	32.857

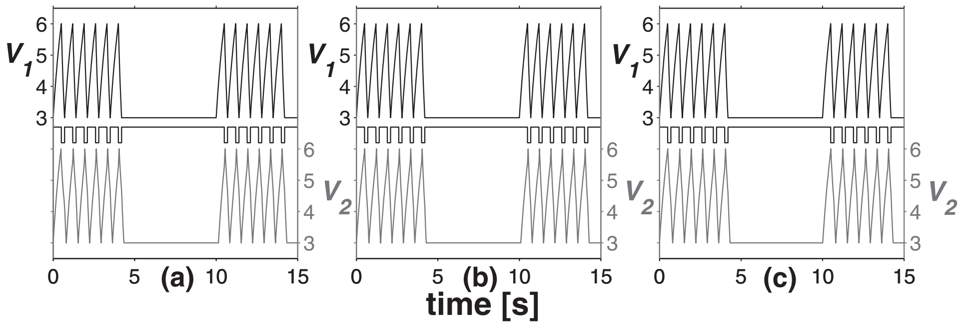

Fig. 4. Dynamic variables V_1 for the master BO and V_2 for the slave BO. The BOs are nonidentical with parameter values for the master: $T_{c01} = 0.500$ s, $T_{d01} = 0.200$ s, and initial condition $\Delta\phi = 0.000$ s; and for the slave: (a) $T_{c02} = 0.480$ s, $T_{d02} = 0.220$ s; (b) $T_{c02} = 0.520$ s, $T_{d02} = 0.180$ s; and (c) $T_{c02} = 0.510$ s, $T_{d02} = 0.210$ s. The coupling strength is $\beta = 8.0$.

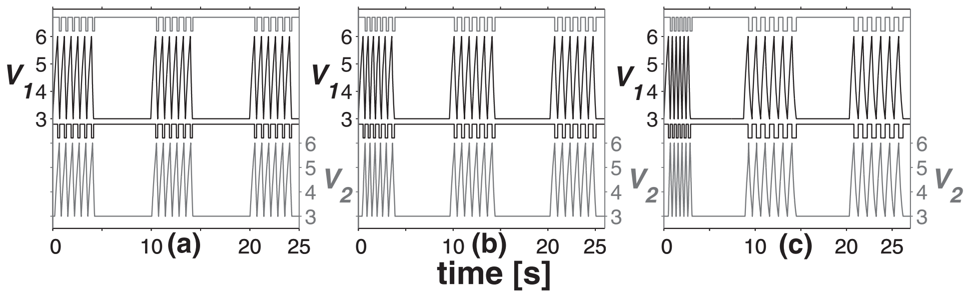

Fig. 5. Dynamic variables V_1 and V_2 for mutually coupled BOs. The BOs are identical with parameter values: $T_{c0} = 0.500$ s, $T_{d0} = 0.200$ s, and coupling strength (a) $\beta = 1.0$, (b) $\beta = 5.0$, and (c) $\beta = 8.0$; with initial conditions $\Delta\phi_1 = 0.000$ s for all cases, and $\Delta\phi_2 = 0.200$ s, 0.400 s and 0.485 s respectively for (a)–(c). Note that the binary variables ε_1 and ε_2 are represented in each panel, in the central and the upper part respectively.

and this process for the BOs occurs chiefly during the first active phase. Another important aspect is that the silent time, as a rule, remains constant.

When working with nonidentical BOs, a first observation is that contrary to the other cases studied until now, synchronization is not manifested by the equality of each intraburst period but by the fact that active phases and/or silent times, and interburst periods are the same for the oscillators. A glance of synchronous behavior of mutually coupled nonidentical BOs is shown in Fig. 6, and a summary of the possibilities in Table 5. In order to illustrate that intraburst periods are different, we consider the first row of Table 5, where the intraburst periods for the BO_1 are: 0.787 s, 0.809 s, 0.801 s, 0.795 s, 0.791 s and 0.789 s, and for the BO_2: 0.802 s, 0.827 s, 0.809 s,

Table 4. Modifications on the charging and discharging time and also on the period of two mutually coupled BOs compared with the "natural" quantities: $T_{c0} = 0.500$ s, $T_{d0} = 0.200$ s and $T_0 = 0.700$ s. Case of identical oscillators.

β	$T_{c1} = T_{c2}$ [s]	$\%\Delta T_c$	$T_{d1} = T_{d2}$ [s]	$\%\Delta T_d$	$T_1 = T_2$ [s]	$\%\Delta T$
1.0	0.500	0.00	0.215	7.50	0.715	2.14
5.0	0.500	0.00	0.309	54.50	0.809	15.57
8.0	0.499	−0.20	0.482	141.00	0.981	40.14

Table 5. Duration of active phases (T_a), silent times (T_s) and the corresponding interburst periods (T_p) for two mutually nonidentical BOs, where the "natural" parameter values for the BO_1 are given by: $T_{c01} = 0.500$ s, $T_{d01} = 0.200$ s; being the symmetrical coupling strength $\beta = 5.0$, and the initial conditions $\Delta\phi_1 =$ 0.000 s, $\Delta\phi_2 =$ 0.300 s.

T_{c02} [s]	T_{d02} [s]	T_{a1} [s]	T_{s1} [s]	T_{a2} [s]	T_{s2} [s]	$T_p = T_{p1} = T_{p2}$ [s]
0.500	0.210	4.722	5.799	4.832	5.739	10.571
0.500	0.190	4.692	5.799	4.632	5.859	10.491
0.510	0.200	4.800	5.799	4.860	5.739	10.599
0.490	0.200	4.800	5.799	4.740	5.859	10.599
0.510	0.190	4.705	5.799	4.705	5.799	10.504
0.490	0.210	4.784	5.799	4.784	5.799	10.583
0.510	0.210	4.719	5.799	4.839	5.679	10.518
0.490	0.190	4.641	5.799	4.521	5.919	10.440

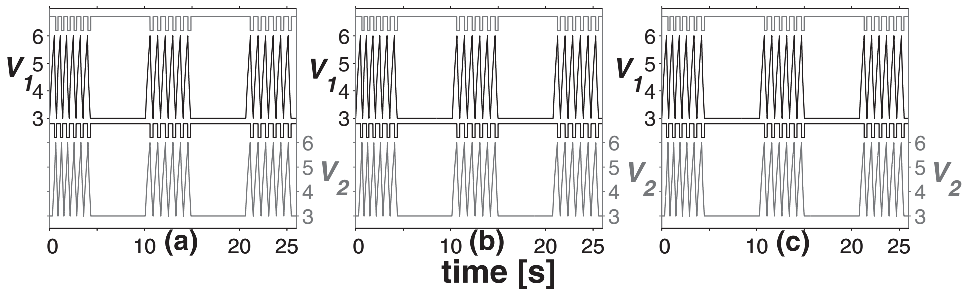

Fig. 6. Dynamic variables V_1 and V_2 for mutually coupled nonidentical BOs with parameter values: $T_{c01} = 0.500$ s, $T_{d01} = 0.200$ s, $\Delta\phi_1 =$ 0.000 s, $\Delta\phi_2 =$ 0.300 s, symmetrical coupling strength $\beta = 5.0$, and (a) $T_{c02} = 0.500$ s and $T_{d02} = 0.210$ s, (b) $T_{c02} = 0.510$ s and $T_{d02} = 0.200$ s, and (c) $T_{c02} = 0.510$ s and $T_{d02} = 0.190$ s. The main feature of synchronization is the fact that interburst periods (T_p) are the same for both BOs; thus, $T_{p1} = T_{p2}$ holds with values 10.571 s, 10.599 s and 10.504 s respectively for (a)–(c). The binary variables ε_1 and ε_2 are represented in each panel, in the central and the upper part respectively.

0.800 s, 0.793 s and 0.791 s. According to the results, we see that synchronization is possible for the BOs even for the nonidentical case.

3.3 Master (NBO)–slave (NBO) configuration

When dealing with identical NBOs, under a master–slave configuration, we observe interesting features such as the evolution of the slave's signal towards the emergence of a delay between master and slave signals corresponding to the active phase of the slave oscillator (see Figs. 7b and c). In the case in which both NBOs have the same initial conditions, there is an almost perfect overlap of the signals and the consequent simultaneous firing of the NBOs, as shown in Fig. 7a. For the purpose of clarifying

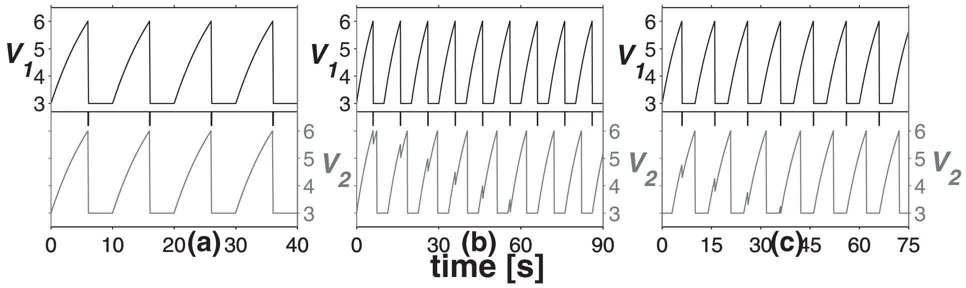

Fig. 7. Dynamic variables V_1 for the master NBO and V_2 for the slave NBO. The NBOs are identical with parameter values: $T_{c0} = 6.000\,\text{s}$, $T_{d0} = 0.100\,\text{s}$, coupling $\beta = 5.0$ and initial condition $\Delta\phi_1 = 0.000\,\text{s}$ and: (a) $\Delta\phi_2 = 0.000\,\text{s}$, (b) $\Delta\phi_2 = 0.010\,\text{s}$and (c) $\Delta\phi_2 = 3.000\,\text{s}$.

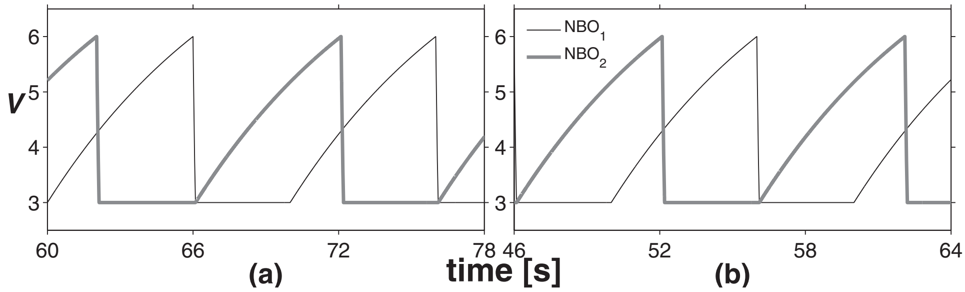

Fig. 8. Representation of both master (NBO_1) and slave (NBO_2) signals corresponding to regions in which a stable behavior is attained of (a) Fig. 7b and (b) Fig. 7c. Note that there is a delay between the signals that is equal to the slave's active phase, and that master silent time follows that of the slave.

what happens as a result of the action of the master on the slave, we represent in Figs. 8a and b, a magnification of a region of Figs. 7b and c, where it is possible to have a better insight on the delay between the master and slave signal that corresponds to the active phase of the slave. Consequently, the NBOs cannot share their firings except when they have the same initial conditions.

When the NBOs are nonidentical, there is still an evolution of the slave towards the situation in which, the silent time of the slave is followed by that of the master as shown in Figs. 9b and c. In this case, the delay between the two signals corresponds to the active phase of the slave as in the case of identical NBOs and similar situations to those shown in Fig. 8 are present. Once again, if the NBOs have the same initial conditions, the signals of both NBOs almost overlap with the consequent coincidence in their firing as shown in Fig. 9a. Another important feature, in this case, is that the slave NBO recovers its "natural" parameter values, i.e., its own frequency. The latter implies that under these circumstances, the master does not impose its frequency to the slave, and consequently, NBOs do not share their firing process.

3.4 Mutually coupled NBOs configuration

When two NBOs are mutually coupled, we observe (see Fig. 10) that the oscillators never share their firing times when they do not have the same initial conditions. Nevertheless, they synchronize due to a sharp increase of their charging times, while, their discharging times remain constant as shown in Table 6. From Figs. 10a and b, we note that, contrary to the case of the master–slave configuration, the silent times

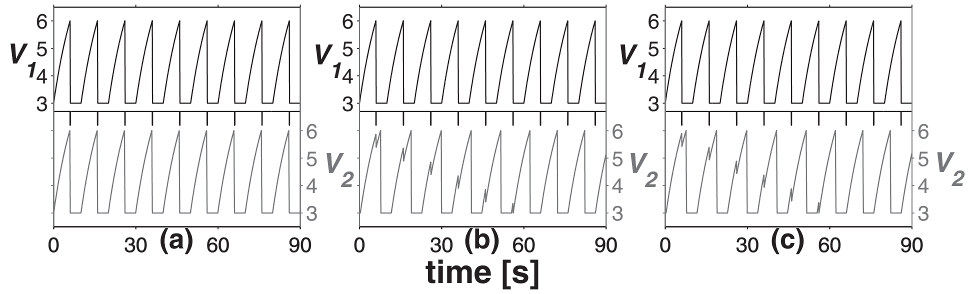

Fig. 9. Dynamic variables V_1 for the master NBO and V_2 for the slave NBO. The NBOs are nonidentical with parameter values: $T_{c01} = 6.000$ s, $T_{d01} = 0.100$ s, coupling $\beta = 5.0$, initial condition $\Delta\phi_1 = 0.000$ s and: (a) $T_{c02} = 5.900$ s, $T_{d02} = 0.200$ s and $\Delta\phi_2 = 0.000$ s, (b) $T_{c02} = 6.100$ s, $T_{d02} = 0.100$ s and $\Delta\phi_2 = 0.300$ s, and (c) $T_{c02} = 6.000$ s, $T_{d01} = 0.200$ s $\Delta\phi_2 = 0.300$ s.

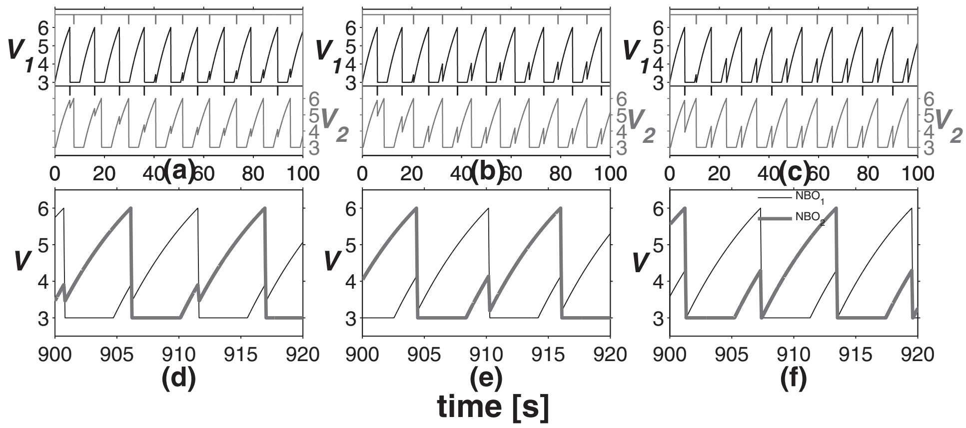

Fig. 10. Dynamic variables V_1 and V_2 for mutually coupled NBOs (NBO_1 and NBO_2 respectively). The NBOs are identical with parameter values: $T_{c0} = 6.000$ s, $T_{d0} = 0.100$ s, initial conditions $\Delta\phi_1 = 0.000$, $\Delta\phi_2 = 0.100$ s and symmetrical coupling strength (a) $\beta = 5.0$, (b) $\beta = 10.0$, and (c) $\beta = 20.0$. The binary variables ε_1 and ε_2 are represented in each panel, in the central and the upper part respectively. (d)–(f) represent the signals in a magnified form for the cases (a)–(c) respectively and when synchronization turns out to be stable.

Table 6. Modifications on the discharging and charging times and also on the period of two mutually coupled NBOs compared with the "natural" quantities: $T_{d0} = 0.100$ s, $T_{c0} = 6.000$ s, and $T_0 = 6.100$ s. Case of identical oscillators.

β	$T_{d1} = T_{d2}$ [s]	$\%\Delta T_d$	$T_{c1} = T_{c2}$ [s]	$\%\Delta T_d$	$T_1 = T_2$ [s]	$\%\Delta T$
5.0	0.100	0.00	6.815	13.58	6.915	13.36
10.0	0.100	0.00	7.627	27.12	7.727	26.67
20.0	0.100	0.00	8.200	36.67	8.300	36.07

of the oscillators do not follow one another. We also observe that the charging times grow markedly as a result of the inhibitory coupling during the charging stages. When the coupling is enough strong, as in Fig. 10f, the charging stage is reset to its baseline.

When considering two mutually coupled nonidentical NBOs, we observe from the examples shown in Fig. 11 a similar behavior than in the other cases related to NBOs, i.e., in general, the NBOs do not fire simultaneously and in some cases, the silent times

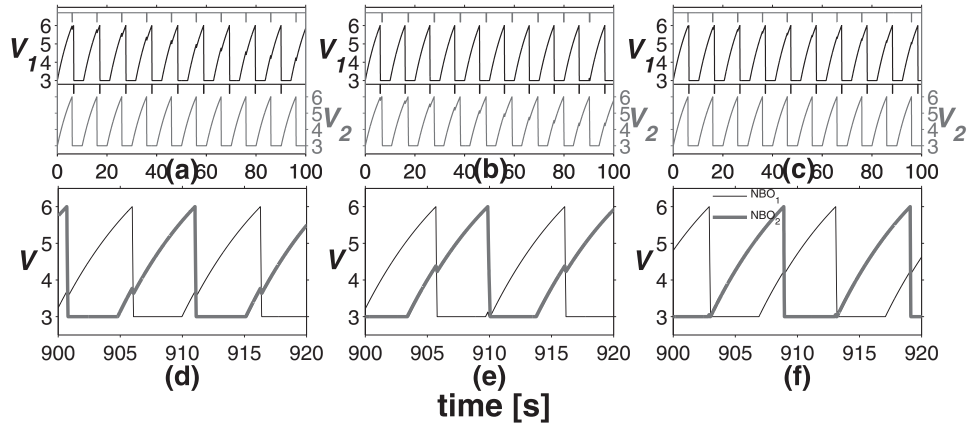

Fig. 11. Dynamic variables V_1 and V_2 for nonidentical mutually coupled NBOs (NBO_1 and NBO_2 respectively) with parameter values: $T_{c01} = 6.000$ s, $T_{d01} = 0.100$ s, initial conditions $\Delta\phi_1 = 0.000$ s, $\Delta\phi_2 = 0.030$ s, $\Delta\phi_2 = 0.100$ s and symmetrical coupling strength $\beta = 2.0$ when (a) $T_{c02} = 5.900$ s, $T_{d02} = 0.100$ s, (b) $T_{c02} = 6.100$ s, $T_{d02} = 0.200$ s and $T_{c02} = 5.900$ s, $T_{d02} = 0.005$ s. The binary variables ε_1 and ε_2 are represented in each panel, in the central and the upper part respectively. (d)–(f) represent the signals in a magnified form for the cases (a)–(c) respectively and when the systems attains a stable situation.

Table 7. Duration of charging and discharging times and active phases for two mutually coupled nonidentical NBOs, where the "natural" parameter values for NBO_1 are given by: $T_{c01} = 6.000$ s, $T_{d01} = 0.100$ s; being the symmetrical coupling strength $\beta = 2.0$, and the initial conditions $\Delta\phi_1 =$ 0.000 s, $\Delta\phi_2 = 0.030$ s.

T_{c02} [s]	T_{d02} [s]	T_{c1} [s]	T_{d1} [s]	T_{a1} [s]	T_{c2} [s]	T_{d2} [s]	T_{a2} [s]
6.000	0.200	6.366	0.100	6.466	6.366	0.200	6.566
6.000	0.050	6.183	0.100	6.283	6.183	0.050	6.233
6.100	0.100	6.321	0.100	6.431	6.431	0.100	6.531
5.900	0.200	6.357	0.100	6.457	6.307	0.200	6.507
6.100	0.005	6.185	0.100	6.285	6.285	0.050	6.335
5.900	0.100	6.330	0.100	6.430	6.213	0.100	6.313
6.100	0.200	6.375	0.100	6.475	6.475	0.200	6.675
5.900	0.050	6.181	0.100	6.281	6.081	0.050	6.131

follow one after the other (Fig. 11c). On the other hand, in a wide range of parameter values, the NBOs' charging times grow significantly, as shown in Table 7. Concerning the discharging times, they preserve their "natural" values when the system stabilizes.

The study of different configurations of NBOs shows that synchronization between them is not a usual feature, and in general, they do not fire simultaneously, except when their initial conditions are the same.

3.5 Master (BO)–slave (NBO) configuration

In Sects. 3.1–3.4, we clearly established the differences between BOs and NBOs by studying several configurations of the same type of oscillators. The results obtained in the aforementioned sections show that BOs and NBOs display different behavior and in this sense, we can affirm that they are strongly dissimilar. Here, we address the study of configurations in which both types of oscillators are present. When a

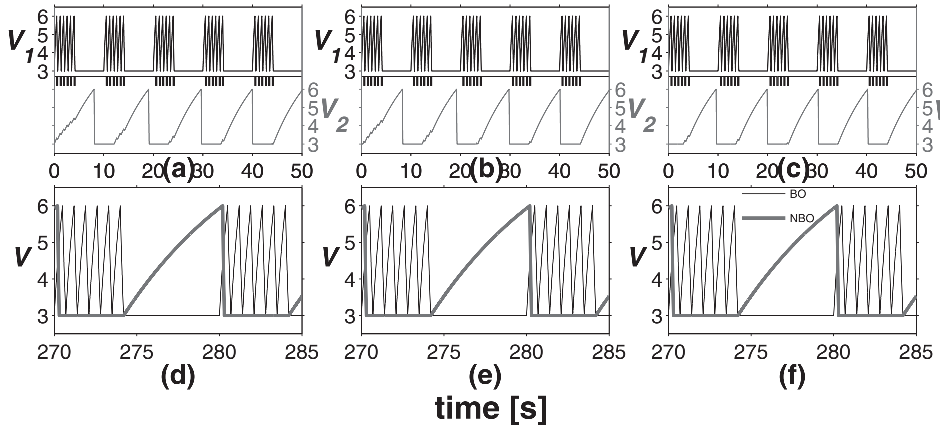

Fig. 12. Dynamic variables V_1 and V_2 corresponding respectively to a BO (master) and an NBO (slave) with parameter values: $T_{c1} = 0.500\,\mathrm{s}$, $T_{d1} = 0.200\,\mathrm{s}$, $T_{c02} = 6.000\,\mathrm{s}$, $T_{d02} = 0.100\,\mathrm{s}$, $\beta = 1.0$, initial conditions $\Delta\phi_1 = 0.000\,\mathrm{s}$, (a) $\Delta\phi_2 = 0.000\,\mathrm{s}$, (b) $\Delta\phi_2 = 0.300\,\mathrm{s}$ and (c) $\Delta\phi_2 = 3.000\,\mathrm{s}$. (d)–(f) representation of the same situations (a)–(c) but when the behavior has attained stability.

BO plays the role of a master acting on an NBO (slave), we observe that for different initial conditions, the BO drives the NBO, producing firstly, an increase of the active phase of the NBO (see Figs. 12a–c), and subsequently, when stabilization occurs, causing that the silent time of the NBO is immediately followed by that of the BO (see Fig. 12d–f). According to the latter results and the NBO's features, it is found that the NBO does not modify its "natural" parameters and it always fires when the active phase of the BO starts.

3.6 Master (NBO)–slave (BO) configuration

When the NBO plays the role of the master and the BO is the slave, we observe that depending on the initial conditions, the NBO can affect or not the dynamics of the BO. The last situation is the most likely as shown in Fig. 13a, where the NBO fires during the silent time of the BO and consequently without any effect on its dynamics. Another situation is presented in Fig. 13b, where the effect of the firing of the NBO only affects the first spike of the BO's active phase. On the contrary, another initial condition leads to a situation in which only the sixth spike of the BO's active phase is affected by the action of the NBO (see Fig. 13c). Owing to the fact that the NBO's discharging time is very short, the action on the BO's dynamics appears to be insubstantial. Nevertheless, it is important to remark that this slight action could turn out to be important when there are several oscillators as stated in [42]. Figs. 13d–f magnify the situations presented in Figs. 13a–c.

3.7 Mutually coupled BO and NBO configuration

We now address our attention to the study of mutually coupled dissimilar oscillators, i.e, BOs and NBOs. Firstly, we consider one BO mutually coupled to an NBO with parameter values $T_{c01} = 0.500\,\mathrm{s}$, $T_{d01} = 0.200\,\mathrm{s}$, $T_{c02} = 6.000\,\mathrm{s}$, $T_{d02} = 0.100\,\mathrm{s}$, initial conditions $\Delta\phi_1 = 0.000\,\mathrm{s}$, $\Delta\phi_2 = 0.000\,\mathrm{s}$, and three different symmetrical coupling strength as show in Fig. 14. The latter shows that for any coupling strength, the

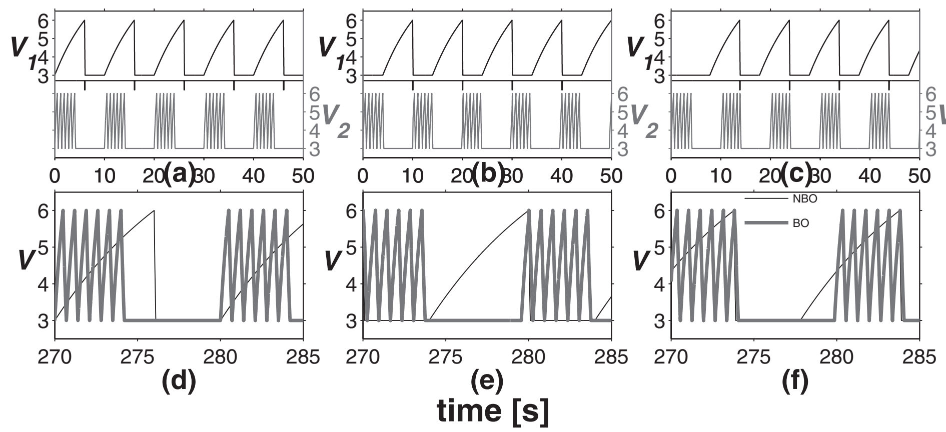

Fig. 13. Dynamic variables V_1 and V_2 corresponding respectively to an NBO (master) and a BO (slave) with parameter values: $T_{c1} = 6.000\,\mathrm{s}$, $T_{d1} = 0.100\,\mathrm{s}$, $T_{c02} = 0.500\,\mathrm{s}$, $T_{d02} = 0.200\,\mathrm{s}$, $\beta = 20.0$, initial conditions $\Delta\phi_2 = 0.000\,\mathrm{s}$, (a) $\Delta\phi_1 = 0.000\,\mathrm{s}$, (b) $\Delta\phi_2 = 4.000\,\mathrm{s}$ and (c) $\Delta\phi_2 = 6.000\,\mathrm{s}$. (d)–(f) representation of the same situations (a)–(c) but when the behavior has attained stability.

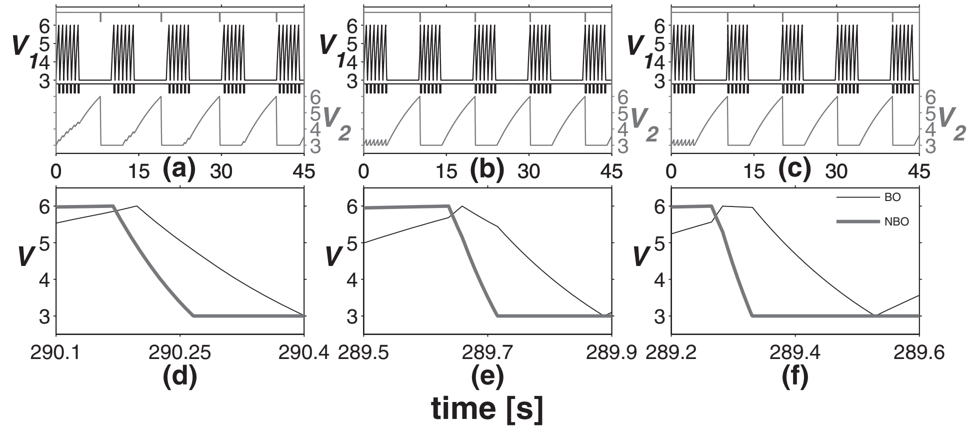

Fig. 14. Dynamic variables V_1 and V_2 corresponding respectively to a BO and an NBO with parameter values: $T_{c01} = 0.500\,\mathrm{s}$, $T_{d01} = 0.200\,\mathrm{s}$, $T_{c02} = 6.000\,\mathrm{s}$, $T_{d02} = 0.100\,\mathrm{s}$, initial conditions $\Delta\phi_1 = 0.000\,\mathrm{s}$, $\Delta\phi_2 = 0.000\,\mathrm{s}$, and symmetrical coupling strength (a) $\beta = 1.0$, (b) $\beta = 10.0$ and (c) $\beta = 20.0$. (d)–(f) representation of the same situations (a)–(c) but focusing on the firings of the NBO and the first spike of the BO when the behavior has attained stability.

NBO ends by firing just before the first firing of the BO. Fig. 14 shows that BO acts considerably on the NBO during the first active phases and forcing the NBO to fire before the first firing of the BO. After this situation is attained, the oscillators only interact during the firing of the NBO and the first firing of the BO. The last produces small changes in the duration of the active phases of both oscillators but the duration of their phrases (T_p) remaining the same as stated in Table 8.

A second situation considers different values of the NBO's charging time with strong coupling $\beta = 20.0$. When the discharging times are less than the value considered in the first situation, we observe that the BO acts on the NBO only during the

Table 8. Duration of active phases (T_a), silent times (T_s) and the corresponding interburst periods (T_p) for a BO and an NBO mutually coupled, where the "natural" parameter values for BO_1 are given by $T_{c01} = 0.500$ s and $T_{d01} = 0.200$ s, and the initial conditions are $\Delta\phi_1 = 0.000$ s, $\Delta\phi_2 = 0.000$ s. Note that nd, OD and v state respectively for not defined, oscillation death and variable.

T_{c02} [s]	T_{d02} [s]	β	T_{a1} [s]	T_{s1} [s]	T_{p1} [s]	T_{a2} [s]	T_{s2} [s]	T_{p2} [s]
6.000	0.100	1.0	5.800	4.199	9.999	3.900	6.099	9.999
6.000	0.100	10.0	5.800	4.180	9.980	3.900	6.080	9.980
6.000	0.100	20.0	5.800	4.167	9.967	3.900	6.067	9.967
5.000	0.100	20.0	5.800	4.200	10.000	4.900	5.100	10.000
6.330	0.100	20.0	5.800	4.200	10.000	nd	OD	OD
7.000	0.100	20.0	5.800	4.200	10.000	nd	OD	OD
7.000	0.100	0.5	5.800	4.200	10.000	2.900	v	v
7.000	0.100	1.0	5.800	4.200	10.000	2.900	v	v
7.000	0.100	2.0	5.800	4.200	10.000	2.900	17.100	20.000

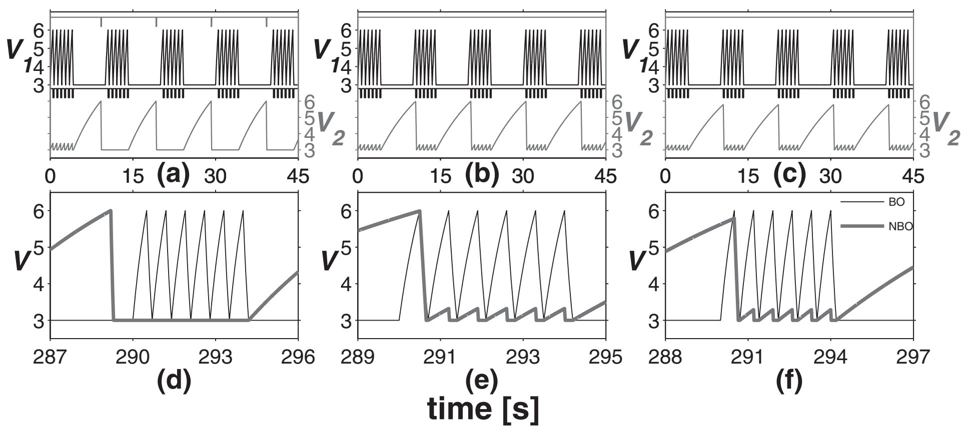

Fig. 15. Dynamic variables V_1 and V_2 corresponding respectively to a BO and an NBO with parameter values: $T_{c01} = 0.500$ s, $T_{d01} = 0.200$ s, $T_{d02} = 0.100$ s, initial conditions $\Delta\phi_1 = 0.000$ s, $\Delta\phi_2 = 0.000$ s, symmetrical coupling strength $\beta = 20.0$ and discharging time for the NBO (a) $T_{c02} = 5.000$ s, (b) $T_{c02} = 6.330$ s and (c) $T_{c02} = 7.000$ s. (d)–(f) representation of the same situations (a)–(c) but focusing on the oscillators' active phases when the behavior has attained stability.

first active phase. Then, there is in a certain way a decoupling between the oscillators because the firings of each oscillator act on the silent times of the other oscillator and consequently without any effect (see Fig. 15a). When the NBO's charging time is greater or equal to $T_{c02} = 6.330$ s, oscillation quenching occurs on the NBO, more concretely, oscillation death, an interesting phenomenon that might arise due to a strong coupling [31] as in the cases shown in Figs. 15b–c. Even though there seems to be an oscillatory behavior on the NBO, actually, oscillation death is manifested by the fact that the NBO's signal cannot attain the upper threshold anymore and as a consequence, it cannot neither fire nor exert any influence on the BO. Other issues got from Figs. 15b–c, are the facts that silent times are not present and also the binary variable ε does neither change; these features are other manifestations of oscillation death.

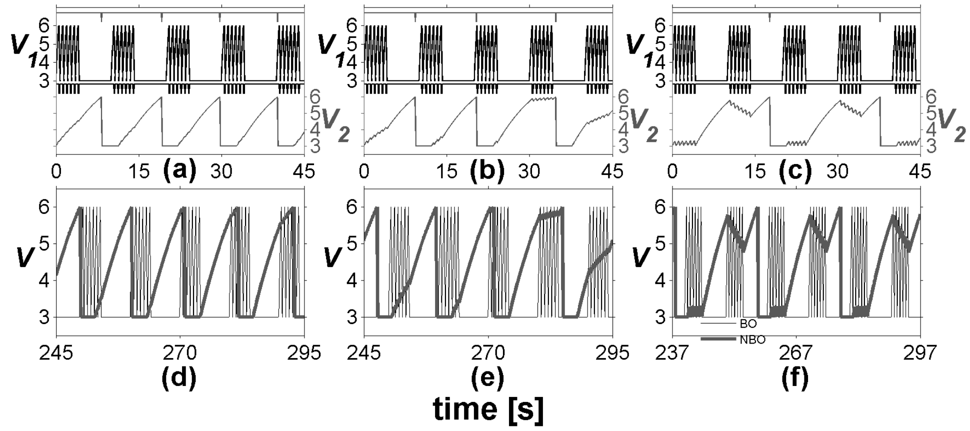

Fig. 16. Dynamic variables V_1 and V_2 corresponding respectively to a BO and an NBO with parameter values: $T_{c01} = 0.500\,\mathrm{s}$, $T_{d01} = 0.200\,\mathrm{s}$, $T_{c02} = 7.000\,\mathrm{s}$, $T_{d02} = 0.100\,\mathrm{s}$, initial conditions $\Delta\phi_1 = 0.000\,\mathrm{s}$, $\Delta\phi_2 = 0.000\,\mathrm{s}$ and symmetrical coupling strength (a) $\beta = 0.5$, (b) $\beta = 1.0$ and (c) $\beta = 2.0$. (d)–(f) representation of the same situations (a)–(c) but focusing on the oscillators' active phases after a certain time.

A final situation is shown in Fig. 16 when the charging time of the NBO is $T_{c02} = 7.000\,\mathrm{s}$ and different coupling strengths are considered. When the coupling strength is weak, the system does not attain stability and the duration of each active phase is not constant (see Figs. 16a–b). In these cases, the NBO can act on the BO and as a result, the BO's active phase becomes slightly shorter as it is stated in Table 8. When the coupling strength increases $\beta = 2.0$, the NBO is periodic but with a much longer active phase (see Fig. 16c). In this case, the system stabilizes and only the BO acts on the NBO, the opposite is not possible. Finally, for greater coupling strengths, we find that oscillation death appears as in Fig. 15c.

The information obtained in this section gives us interesting information in what concerns the behavior of mutually coupled dissimilar oscillators. It is remarkable the fact that oscillation death can occur with the increase of coupling strength and it is also noticeable that for certain situations, the interburst period might be the same for both oscillators.

3.8 Two NBOs and one BO mutually coupled configuration

When there are two NBOs and one BO mutual and symmetrically coupled ($N = 3$), the behavior of the system depends strongly on the coupling strength as shown in Fig. 17, where the evolution of the binary variable ε is depicted for three different values of the coupling strength. In Figs. 17a–b, the system does not stabilize and the NBOs fire at different times. On the other hand, in Fig. 17c, the system stabilizes very quickly to a situation in which the two NBOs fire simultaneously and just before the beginning of an active phase of the BO. The latter signifies that the BO plays a stabilizing role on the NBOs, i.e., it is possible to control the NBOs by means of a BO. The results could be extended to a system composed of several NBOs which can be identical or nonidentical, with or without the same coupling strength.

The results shown in Fig. 17c, indicate that the NBOs might be controlled by the BO. Thus, it is possible to address a study about control issues. Other possibilities such as the fact that only the BO can influence the NBOs (a situation similar to master–slave) deserve a deeper study on how a BO controls NBOs.

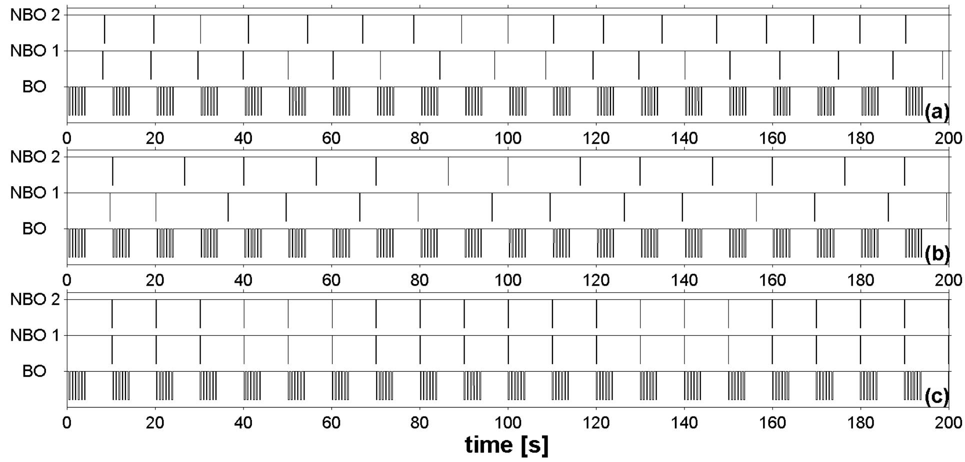

Fig. 17. Evolution of the binary variables ε when parameter values are for the BO $T_{c01} = 0.500\,\text{s}$, $T_{d01} = 0.200\,\text{s}$, and for the NBOs $T_{c0} = 6.000\,\text{s}$, and $T_{d0} = 0.100\,\text{s}$, whereas, the initial conditions are $\Delta\phi_1 = 0.000\,\text{s}$, $\Delta\phi_2 = 0.100\,\text{s}$ and $\Delta\phi_2 = 0.300\,\text{s}$; the coupling strength is symmetrical with values (a) $\beta = 1.0$, (b) $\beta = 2.0$ and (c) $\beta = 3.0$.

3.9 Two or more BOs and one NBO mutually coupled configuration ($N \geq 3$)

Now, when considering two BOs mutually coupled to an NBO, we see that both BOs firstly produce oscillation death on the NBO and then when the BOs synchronize, the NBO recovers its oscillatory behavior. The aforementioned phenomenon occurs very quickly as shown in Fig. 18a. This result is very important because it explains the so-called response to synchronization, a phenomenon occurring in some species of fireflies as it has been stated in Sect. 2. In Fig. 18b are represented the signals from a system constituted of 5 BOs and 3 NBOs. Again, the nonsynchronous BOs provoke oscillation death on the NBOs. Then, a sporadic synchronization of BOs induces simultaneous firing of the NBOs. The desynchronization of the BOs follows the sporadic synchronization until finally, stable synchronization is achieved and consequently with the simultaneous response of the NBOs. It is important to mention that the examples shown in Fig. 18 are characterized by the fact that the oscillators are nonidentical, the coupling strength is symmetrical but not the same for each pair of oscillators, and the initial conditions are randomly chosen.

4 Discussion

From the results obtained in Sect. 3, we must point out that the study of each of the possible configurations, gives us important information on the individual behavior of the oscillators and also about the sets of coupled oscillators. The BOs and the NBOs are dissimilar oscillators not only by the fact that their signals are quite different but also by their reactions to stimuli (the firing of other oscillators). The exhaustive study carried out on both type of oscillators allowed us to understand the individual behavior of these oscillators. Furthermore, we got the essentials to figure out the collective behavior of populations of mutually coupled oscillators. It is important to remark that a set of BOs could easily achieve synchronization, manifested by the sharing of their firings. On the contrary, the individuals of a set of NBOs, in general, cannot fire simultaneously. Two phenomena arise in mingled populations of BOs and NBOs: firstly, the possibility to control the oscillatory behavior of the NBOs' set by means

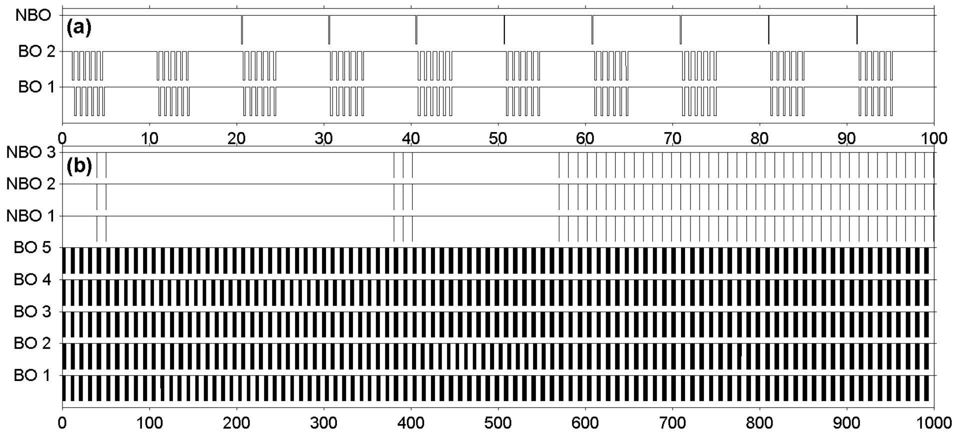

Fig. 18. Evolution of the binary variables ε for a set of BOs and NBOs, being the oscillators of each group nonidentical, the coupling strength symmetrical but not the same for each pair of oscillators, and the initial conditions chosen randomly. (a) Two BOs and one NBO. (b) Five BOs and three NBOs.

of a single BO, and secondly, the emergence of response to synchronization. The response to synchronization occurs after a first process of oscillation death provoked in the NBOs' set as a result of desynchronized BOs. Nevertheless, as BOs synchronize, oscillation death on NBOs is suppressed, i.e, they recover their oscillatory behavior, triggering off simultaneous firing or well-defined firing patterns. From a dynamical systems theory point of view, it is clear that the emergent phenomena studied in this work, are a consequence of the oscillators' features that can give rise to multistability through bifurcations according to their values, especially when dealing with populations $N \geq 3$. The different attained stable regimes depend on the parameters, such as charging and discharging times, interburst period, coupling strength and also initial conditions. The tuning of the above mentioned quantities could lead to different response patterns and also affect the transients in particular when $N \geq 3$. After the detailed study performed in Sect. 3, we can summarize the most important issues in Table 9.

5 Conclusions and perspectives

We have reviewed exhaustively the dynamical aspects of dissimilar oscillators involved in the phenomenon of response to synchronization. The analysis allowed us to unravel the primary mechanisms leading to synchronization response. Firstly, the nonsynchronized BOs induce oscillation death in the NBO(s). Secondly, when the BOs partially synchronize, they induce a sporadic and/or partial response in the NBO(s) (the oscillatory behavior is sporadically recovered). Finally, when the BOs are completely synchronized, the response of the NBO(s) is permanent (the oscillatory behavior is completely recovered). Another interesting phenomenon which was found is pertinent to the possibility to control a population of nonsynchronous oscillators by means of an oscillator that might induce simultaneous firings in the individuals of the aforementioned population. In all cases, parameter values, coupling strength and even initial conditions are important in what concerns the stable state and the transient, i.e., the systems studied in this work exhibit multistability. When dealing with functional systems, e.g., population of fireflies or neurons, the response to synchronization might

Table 9. Summary of the most important results concerning coupled BOs and/or NBOs.

Type of oscillator	Type of oscillator	Coupling configuration	Oscillators' relationship	Main features of the resulting behavior
BO	BO	master–slave	identical	Sharp decline in T_c. Strong increase in T_d. Constant active phase.
BO	BO	master–slave	nonidentical	Sharp decline in T_c. Strong increase in T_d. Constant active phase.
BO	BO	mutually coupled	identical	Synchronization is easily attained.
BO	BO	mutually coupled	nonidentical	Synchronization is manifested by the equality of interburst periods.
NBO	NBO	master–slave	identical	Silent time of the slave is followed by the silent time of the master. They do not fire together.
NBO	NBO	master–slave	nonidentical	Master does not impose its frequency to the slave. They do not fire together.
NBO	NBO	mutually coupled	identical	Strong increase in their charging times T_c. Synchronization with a phase delay. They do not fire together.
NBO	NBO	mutually coupled	nonidentical	Strong increase in their charging times T_c. Synchronization is not very common. They do not fire together.
BO	NBO	mutually coupled		Strong increase in NBO's T_c. Strong coupling can produce oscillation death on NBO.
NBOs	BO	mutually coupled		Strong increase in NBOs' T_c. Strong coupling can produce oscillation death on NBO. BO can control NBOs which can fire simultaneously.
BOs	NBOs	mutually coupled		Strong increase in NBO's T_c. Strong coupling can produce oscillation death on NBO. Synchronization of BOs is followed by the simultaneous firing of NBOs. Response to synchronization.

be explained in terms of two dissimilar groups (for instance males and females). The individuals of one of the groups when emitting disordered signals, inhibit the oscillatory behavior of the individuals of the other group. Then, when the individuals of the first group synchronize, the oscillatory behavior of the individuals of the second group is recovered. The firings of the individuals of the second group might be seen as responses to synchronization of the individuals of the first group. In summary, two interesting phenomena have been studied here: control of firings and response to synchronization. Based on the features stated in this work that have clarified both individual and collective behavior of pulse-coupled dissimilar oscillators, it is possible to deeply study the collective behavior of these interacting dissimilar oscillators both from theoretical and experimental point of view, especially when the sets are composed by a considerable number of oscillators. The latter could contribute to a better understanding of systems that exhibit the phenomenon of response to synchronization, viz. fireflies, neurons, and possibly other animals and other type of cells. We are currently analyzing the role of network topologies, as studied in [48], for the systems described in this work.

G.M.R.A. is indebted to the members of the staff of PIK-Potsdam for their kind hospitality. J.K. acknowledges IRTG 1740 (DFG). We are grateful to S. Depickère for the careful reading of the manuscript.

References

1. I. Blekhman, *Vibrational Mechanics: Nonlinear Dynamic Effects, General Approach, Applications* (World Scientific, Singapore, 2000)
2. I.I. Blekhman, *Synchronization in Science and Technology* (ASME Press, New York, 1988)
3. A. Pikovsky, M. Rosenblum, J. Kurths, *Synchronization: a Universal Concept in Nonlinear Sciences* (Cambridge University Press, New York, 2001)
4. S. Boccaletti, L.M. Pecora, A. Pelaez, Phys. Rev. E **63**, 066219 (2001)
5. H. Nijmeije, A. Rodriguez-Angeles, *Synchronization of Mechanical Systems* (World Scientific, Singapore, 2003)
6. A. Stefanski, *Determining Thresholds of Complete Synchronization, and Application* (World Scientific, 2009)
7. H. Fukuda, H. Morimura, S. Kai, Physica D **205**, 80 (2005)
8. M. Lara-Aparicio, C. Barriga-Montoya, P. Padilla-Longoria, B. Fuentes-Pardo, Math. Biosci. Eng. **11**, 317 (2014)
9. J. Gonzalez-Miranda, *Synchronization And Control Of Chaos: An Introduction For Scientists And Engineers* (Imperial College Press, London, 2004)
10. X.B. Lu, B.Z. Qin, *Synchronization in Complex Networks* (Nova Science, New York, 2011)
11. A.E. Motter, S.A. Myers, M. Anghel, T. Nishikawa, Nat. Phys. **9**, 191 (2013)
12. G. Grinstein, R. Linsker, P. Natl. Acad. Sci. USA **102**, 9948 (2005)
13. W. Zhou, J. Yang, L. Zhou, D. Tong, *Stability and Synchronization Control of Stochastic Neural Networks* (Springer, Berlin, 2016)
14. L. Kocarev, *Consensus and Synchronization in Complex Networks* (Springer, Berlin, 2013)
15. R. Bader, *Nonlinearities and Synchronization in Musical Acoustics and Music Psychology* (Springer, Heidelberg, 2013)
16. S. Boccaletti, J. Kurths, G. Osipov, D.L. Valladares, C.S. Zhou, Phys. Rep. **366**, 1 (2002)
17. S.C. Manrubia, A.S. Mikhailov, D.H. Zanette, *Emergence of Dynamical Order* (World Scientific, Singapore, 2004)
18. S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, D.U. Hwang, Phys. Rep. **424**, 175 (2006)
19. A. Balanov, N. Janson, D. Postnov, O. Sosnovtseva, *Synchronization: From Simple to Complex* (Springer, Berlin, 2007)
20. S. Boccaletti, *The Synchronized Dynamics of Complex Systems* (Elsevier, New York, 2008)

21. J.A. Acebron, L.L. Bonilla, C.J.P. Vicente, F. Ritort, R. Spigler, Rev. Mod. Phys. **77**, 137 (2005)
22. F.A. Rodrigues, T.K.D.M. Peron, P. Ji, J. Kurths, Phys. Rep. **610**, 1 (2016)
23. Y. Kuramoto, Physica D **50**, 15 (1991)
24. S. Bottani, Phys. Rev. E **54**, 2334 (1996)
25. G.M. Ramírez Ávila, J. Kurths, J.L. Guisset, J.L. Deneubourg, Eur. Phys. J. Special Topics **223**, 2759 (2014)
26. A.N. Pisarchik, U. Feudel, Phys. Rep. **540**, 167 (2014)
27. W. Zou, D.V. Senthilkumar, J. Duan, J. Kurths, Phys. Rev. E **90**, 032906 (2014)
28. F.M. Atay, Physica D **183**, 1 (2003)
29. T. Banerjee, D. Biswas, Chaos **23**, 043101 (2013)
30. A. Koseska, E. Volkov, J. Kurths, Chaos **20**, 023132 (2010)
31. A. Koseska, E. Volkov, J. Kurths, Phys. Rep. **531**, 173 (2013)
32. A. Koseska, E. Volkov, J. Kurths, Phys. Rev. Lett. **111**, 024103 (2013)
33. J. Buck, E. Buck, Science **159**, 1319 (1968)
34. J. Copeland, A. Moiseff, J. Insect Physiol. **43**, 965 (1997)
35. A. Moiseff, J. Copeland, J. Insect Behav. **13**, 597 (2000)
36. N. Ohba, Integr. Comp. Biol. **44**, 225 (2004)
37. A.T. Winfree, J. Theor. Biol. **16**, 15 (1967)
38. R.E. Mirollo, S.H. Strogatz, SIAM J. Appl. Math. **50**, 1645 (1990)
39. G.M. Ramírez Ávila, J.L. Guisset, J.L. Deneubourg, Physica D **182**, 254 (2003)
40. J. Buck, J.F. Case, Biol. Bull. **121**, 234 (1961)
41. A. Moiseff, J. Copeland, Science **329**, 181 (2010)
42. G.M. Ramírez Ávila, J.L. Deneubourg, J.L. Guisset, N. Wessel, J. Kurths, Europhys. Lett. **94**, 60007 (2011)
43. T. Buschmann, A. Ewald, A. von Twickel, A. Bschges, Bioinspir. Biomim. **10**, 041001 (2015)
44. B.V.C. Martins, G. Brunetto, F. Sato, V.R. Coluci, D.S. Galvão, Chem. Phys. Lett. **453**, 290 (2008)
45. J. Buck, E. Buck, Am. Nat. **112**, 471 (1978)
46. W. Woods Jr, H. Hendrickson, J. Mason, S. Lewis, Am. Nat. **170**, 702 (2007)
47. E. Izhikevich, *Dynamical Systems in Neuroscience: The Geometry of Excitability and Bursting* (MIT Press, Cambridge, 2007)
48. P. Schultz, T. Peron, D. Eroglu, T. Stemler, G.M. Ramírez Ávila, F.A. Rodrigues, J. Kurths, Phys. Rev. E **93**, 062211 (2016)